On the Mean Past Lifetime of the Components of a Parallel System

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Abstract

In the study of reliability of the technical systems and subsystems, parallel systems play a very important role. In the present paper we consider a parallel system consisting of n identical components with independent lifetimes having a common distribution function F. It is assumed that at time t the system has failed. Under these conditions, we obtain the mean past lifetime (MPL) of the components of the system. Some properties of MPL are studied. It is shown that the underlying distribution function F can be recovered from the proposed MPL. Also, a comparison between two parallel systems are made, based on their MPL's, in the case where the components of the system are ordered in terms of reversed hazard rate. Finally a characterization of uniform distribution is given based on MPL.

1 Introduction

Many systems or subsystems have a k-out-of-n structure. A k-out-of-n system is a system consisting of n components (usually the same) and functions if and only if at least k out of n components are operating. Hence, such system fails if $(n-k+1)$ or more of its components fail. Important particular cases of k-out-of-n systems are parallel systems and series systems corresponding to $k=1$ and $k=n$, respectively. Let $T_1, ..., T_n$ denote the lifetimes of the components of the system and assume that the $T_i^{'}$s, $i=1,...,n$ are independent and have a common distribution F. Let also $T_{1:n}, ..., T_{n:n}$ be the order statistics corresponding to $T_i^{'}$s. Assuming that all components of the k-out-of-n system are starting to work at the time $t=0$, then the $(n-k+1)$th order statistics, i.e. $T_{n-k+1:n}$ denotes the lifetime of the system. In particular case when $i=n$, $T_{n:n}$ denotes the lifetime of a parallel system.

In reliability theory, in the study and modeling of the lifetime of a living organism there have been proposed some measures such as the hazard rate and the mean residual life function. The hazard rate and the mean residual life function of T at time t, which we denote them respectively by $r(t)$ and $m(t)$, are defined as follows:

$$r(t) = \frac{f(t)}{\bar{F}(t)}$$

and

$$m(t) = E(T-t|T \geq t) = \frac{\int_t^{\infty} \bar{F}(x)dx}{\bar{F}(t)}$$

provided that $\bar{F}(t) > 0$, where $\bar{F} = 1-F$ denotes the reliability function. Many properties and applications of these measures are obtained in the literature (we refer to Kotz and Shnabhag (1980) and Guess and Proschan (1988) and references therein).

For the parallel system described above, as the lifetime of the system is $T_{n:n}$, the mean residual life function is defined to be

$$m(t) = E(T_{n:n} - t|T_{n:n} \geq t).$$

Recently, Asadi and Bairamov (2003), under the condition that $T_{r:n} > t$, i.e. $(n-r+1)$, $r=1,2,...,n$, components of the system are still working, have proposed a new definition for the mean residual life function of the system as follows and obtained several properties of it:

$$m_n^r(t) = E(T_{n:n} - t | T_{r:n} \geq t), \quad r = 1, 2, ..., n, \tag{1}$$

(see also, Bairamov et al (2002)).

In the present paper we consider a parallel system. On the basis of the structure of parallel systems, when a component with lifetime $T_{r:n}$, $r = 1, 2, ..., n-1$, fails the system is continuing to work until $T_{n:n}$ fails. In fact, the system can be considered as a black box in the sense that the exact failure time of $T_{r:n}$ in it is unknown. Motivated by this, we assume that at time t the system is not working and in fact it has failed at time t or sometime before time t. We rise the following question: What is the mean time elapsed since the r-th component failure? This problem is related to the problem of analyzing so called *autopsy* data, i.e. information obtained by examining the component states of a failed system. For more details on this, we refer to Meilijson (1981), Gȧsemyr and Natvig (1998) and Gȧsemyr and Natvig (2001), among others.

Let us define $\phi_n^r(t) = t - T_{r:n} | T_{n:n} \leq t$, $t > 0$, $r = 1, ..., n$,. This conditional random variable shows, in fact, the time that has passed from the failure of the component with lifetime $T_{r:n}$ in the system given that the system has failed at or before time t. If we denote the expectation of $\phi_n^r(t)$ by $M_n^r(t)$, i.e.

$$M_n^r(t) = E(\phi_n^r(t)), \quad n \geq 1, \ r = 1, ..., n.$$

then, $M_n^r(t)$ measures the MPL from the failure of the component with lifetime $T_{r:n}$ given that the system has lifetime less than or equal to t. It should be mentioned here that $M_n^r(t)$ is the MPL of $T_{r:n}$ in the system level while $E(t - T_{r:n} | T_{r:n} \leq t)$ denotes the MPL of $T_{r:n}$ in the component level.

2 The mean past lifetime of the components of a parallel system

Consider a parallel system with n non-negative independent components having a common continuous distribution function F with left extremity and right extremity $a = inf\{t; F(t) > 0\} \geq 0$ and $b = sup\{t; F(t) < 1\} \leq \infty$, respectively. In the following we first obtain the distribution of $\phi_n^r(t)$. Let, for $x < t$, $x, t \in (a, b)$, $R(x|t)$ denote the reliability function of $\phi_n^r(t)$. Then, for $r = 1, ..., n$, we have

$$R(x|t) = \sum_{i=r}^{n} \binom{n}{i} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \left(\frac{F(t-x)}{F(t)} \right)^{i+j}. \tag{2}$$

Using the survival function given in (2), $M_n^r(t)$ can be obtain as follows:

$$M_n^r(t) = E(\phi_n^r(t)) = \sum_{i=r}^{n} \binom{n}{i} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} M_{i+j}(t). \tag{3}$$

where

$$M_k(t) = \frac{\int_0^t F^k(u) du}{F^k(t)}.$$

$M_k(t)$ is in fact $E(t - T_{k:n} | T_{k:n} \leq t)$.

Remark 2.1 On the basis of Equation (2) $M_n^r(t)$ can also be represented as

$$M_n^r(t) = \int_0^t P(Y_t^x \geq r) dx \tag{4}$$

where Y_t^x is a random variable distributed as $Binomial(n, \theta_t(x))$, with $\theta_t(x) = \frac{F(x)}{F(t)}$, $x < t$.

Remark 2.2 It is obvious from (3) that the following recurrence relation holds for MPL $M_n^r(t)$.

$$M_n^r(t) = \binom{n}{r} \sum_{j=0}^{n-r} (-1)^j \binom{n-r}{j} M_{r+j}(t) + M_n^{r+1}(t). \tag{5}$$

Remark 2.3 $M_n^r(t)$ is a decreasing function of r, $r = 1, ..., n$. This follows easily on noting that

$$\begin{aligned} M_n^r(t) - M_n^{r+1}(t) &= E[(t - T_{r:n}) - (t - T_{r+1:n} | T_{n:n} \le t)] \\ &= E[T_{r+1:n} - T_{r:n} | T_{n:n} \le t] \ge 0. \end{aligned} \tag{6}$$

The right hand side is the expectation of a conditional spacing and denotes the average time between the $(r+1)$th and the rth failures in the system given that the system has already failed at or before time t.

Remark 2.4 On noting that $E(T_{r:n} | T_{n:n} \le t) \ge 0$, for all $t \in (a, b)$ we can easily see that $M_n^r(t) \le t$.

Example 2.5 Let $T_i^{'}$s $i = 1, ..., n$, $n \ge 1$, be independent and suppose that they are distributed as exponential with mean 1. Then

$$R(x|t) = \sum_{i=r}^{n} \binom{n}{i} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} (\frac{e^t - e^x}{e^t - 1})^{i+j}, \quad x < t, \ t > 0.$$

Using this it can be shown that for $r = 1, ..., n$,

$$M_n^r(t) = \sum_{i=r}^{n} \binom{n}{i} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \frac{\sum_{k=0}^{i+j} (-1)^k \binom{i+j}{k} \frac{1}{k} (1 - e^{-kt})}{(1 - e^{-t})^{i+j}}.$$

Example 2.6 Let $T_i^{'}$s $i = 1, ..., n$ $n \ge 1$, be independent and suppose that they are distributed as uniform on $(0, \ \theta)$, $\theta > 0$. Then

$$R(x|t) = \sum_{i=r}^{n} \binom{n}{i} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} (1 - \frac{x}{t})^{i+j}, \quad x < t.$$

Also, it can be easily seen in this case that $M_n^r(t)$ is equal to

$$M_n^r(t) = \frac{n - r + 1}{n + 1} t \tag{7}$$

From this it can be deduced that

$$M_n^r(t) - M_n^{r+1}(t) = \frac{t}{n+1}. \quad , r = 1, 2, ..., n-1,$$

which shows that the difference between the MPL's of rth and $(r+1)$th components does not depend on r.

3 Some properties of $M_n^r(t)$

In this section we study some properties of MPL $M_n^r(t)$. First, in the following theorem we show that $M_n^r(t)$ is an increasing function of n.

Theorem 3.1 *For fixed value of* r, $M_n^r(t)$ *is an increasing function of* n, $n = 1, 2, ...$ *Moreover, if* F *is an absolutely continuous distribution function* $M_n^r(t)$ *is a strictly increasing function of* n.

In the following theorem we show that that the distribution function F can be recovered from $M_n^r(t)$ and $M_{n-1}^r(t)$.

Theorem 3.2 *Let the components of the system have a common absolutely continuous distribution function F with left and right extremities a and b defined as above. Let also f denote the density function of F. Then the distribution function F can be represented in terms of $M_n^r(t)$ as follows:*

$$F(t) = \exp\{-\frac{1}{n}\int_t^b \frac{1-\frac{dM_n^r(x)}{dx}}{M_n^r(x)-M_{n-1}^r(x)}dx\} \qquad t\in(a,b) \qquad n\geq 1,\ r=1,2...,n,, \tag{8}$$

with $M_0^r(t)=0$ for $n=1$.

The following theorem gives a comparison between the MPL's of two parallel systems.

Theorem 3.3 *Consider two parallel systems S_1 and S_2 each consisting of n independent and identical components. Let the components of $S_1(S_2)$ have the common distribution $F(G)$ and reversed hazard functions $r_F(r_G)$, respectively. If $r_F(t)\geq r_G(t)$, $t\in(a,b)$, then*

$$M_n^r(t)\leq K_n^r(t), \quad r=1,2,...,n$$

where $M_n^r(t)$ and $K_n^r(t)$ denote the MPL's of S_1 and S_2, respectively.

4 Some characterizations of uniform distribution

In the following we give some characterizations on uniform distribution based on $\phi_n^1(t)$ and $M_n^1(t)$.

Theorem 4.1 *Let $T_1,...,T_n$ be independent non-negative random variables from a continuous distribution function F. Let also $T_{1:n},...,T_{n:n}$ denote the corresponding order statistics. Then*

$$t-T_{1:n}|T_{n:n}\leq t \overset{d}{=} T_{n:n}|T_{n:n}\leq t, \quad t>0, \tag{9}$$

if and only if for some $\theta>0$ F is uniform on $(0,\ \theta)$ where d stands for distribution.

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Theorem 4.2 *Let $T_1,...,T_n$ be independent non-negative random variables from a continuous distribution function F, such that $\lim \frac{F(t)}{t}$ exists as $t\to 0$. Then*

$$E(t-T_{1:n}|T_{n:n}\leq t) = E(T_{n:n}|T_{n:n}\leq t), \quad t>0, \tag{10}$$

if and only if for some $\theta>0$, F is uniform on $(0,\ \theta)$.

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